An Attempt to Derive the ε Equation from a Two-Point Closure

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ABSTRACT

The goal of this paper is to derive the equation for the turbulence dissipation rate ε for a shear-driven flow. In 1961, Davydov used a one-point closure model to derive the ε equation from first principles but the final result contained undetermined terms and thus lacked predictive power. Both in 1987 (Schiestel) and in 2001 (Rubinstein and Zhou), attempts were made to derive the ε equation from first principles using a two-point closure, but their methods relied on a phenomenological assumption. The standard practice has thus been to employ a heuristic form of the ε equation that contains three empirical ingredients: *two constants*, $c_{1,\varepsilon}$ and $c_{2,\varepsilon}$, and a diffusion term D_ε. In this work, a two-point closure is employed, yielding the following results: 1) the empirical constants get replaced by c_1, c_2, which are now functions of K and ε; 2) c_1 and c_2 are not independent because a general relation between the two that are valid for any K and ε are derived; 3) c_1, c_2 become constant with values close to the empirical values $c_{1,\varepsilon}$, $c_{2,\varepsilon}$ (i.e., homogenous flows); and 4) the empirical form of the diffusion term D_ε is no longer needed because it gets substituted by the K–ε dependence of c_1, c_2, which plays the role of the diffusion, together with the diffusion of the turbulent kinetic energy D_K, which now enters the new ε equation (i.e., inhomogeneous flows). Thus, the three empirical ingredients $c_{1,\varepsilon}$, $c_{2,\varepsilon}$, D_ε are replaced by a single function $c_1(K, \varepsilon)$ or $c_2(K, \varepsilon)$, plus a D_K term. Three tests of the new equation for ε are presented: one concerning channel flow and two concerning the shear-driven planetary boundary layer (PBL).

1. Introduction

The dynamic equations for the kinetic energy K and the dissipation ε are two basic equations in turbulence. While the first equation is exact, the second is ''entirely empirical,'' as Pope (2000) has described the commonly used form of it (P stands for production and D_ε is the diffusion):

$$\frac{\partial \varepsilon}{\partial t} + D_\varepsilon = \frac{\varepsilon}{K}(c_{1,\varepsilon}P - c_{2,\varepsilon}\varepsilon). \tag{1}$$

Although an exact equation for ε was derived by Davydov (1961) using a one-point closure, the final result contained unknown terms and thus had no predictive power. Attempts to use a two-point closure using the definition

$$\varepsilon = 2\nu \int_0^\infty k^2 E(k)\, dk \tag{2}$$

face different types of difficulties. This was emphasized by Pope (2000), who pointed out that (2) interprets ε as pertaining to the dissipation range whereas he suggests it is more physical to interpret ε as the energy flow rate in the cascade process, which is determined by large scales that are independent of viscosity. The choice of small- versus

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large-scale approach has important implications because any approach based on (2) is not likely to be successful since no known two-point closure exists that is capable of providing a reliable energy spectrum in the high-wavenumber regime where the integrand peaks. The large-scale ''option'' is more hopeful since the low-k infrared (IR) regime has been amenable to more successful predictions than the high-k regime. Different studies of the IR regime (e.g., Batchelor 1970; Chasnov 1995; Reynolds 1987) have suggested a functional dependence of the type $E(k) \sim k^s$ with $s = 2, 4$. A key advantage of these results is that they are almost model independent, as the work of Saffman (1967) showed. In that paper, one can also find a list of references of previous work on the IR spectrum.

The availability of infrared spectra is, however, not sufficient to derive an equation for ε. The first reason is that one needs to continue the spectrum in the k region after the IR regime; this does not seem to be a serious conceptual or practical problem since use of a Kolmogorov spectrum for $E(k)$ and a Lumley spectrum (Lumley 1967; Pope 2000) for shear are reasonable assumptions. Adding to the IR spectrum a Kolmogorov spectrum was the approach pursued by Schiestel (1987) and Rubinstein and Zhou (2001), who arrived at a final equation for ε under the following conditions: (a) since the number of equations was less than the number of unknowns, the authors had to adopt an empirical relation, Eq. (12) below, a procedure that Rubinstein and Zhou (2001) called ''another way of stating the final result'' and (b) the final result was only valid for homogeneous turbulence corresponding to $D_\varepsilon = 0$ in Eq. (1). It must nonetheless be stressed that they predicted the values of the constants $c_{1,\varepsilon} = 1.5$, $c_{2,\varepsilon} = 11/6$, which were remarkably close to $c_{1,\varepsilon} = 1.44$, $c_{2,\varepsilon} = 1.92$ cited by Pope (2000) as ''compromise'' values (the values of $c_{2,\varepsilon}$ determined from freely decaying turbulence are 1.87, 1.77, and 1.69). The conclusion one may legitimately draw from these results is that the approach used in Schiestel (1987) and Rubinstein and Zhou (2001) is promising but incomplete in two aspects, which we will try to find a way around.

The first task is to try to avoid the empirical relation (12) below used in Schiestel (1987) and Rubinstein and Zhou (2001) by replacing it with one derived on a theoretical basis, while the second is to avoid the homogeneous, zero-diffusion approximation $D_\varepsilon = 0$. Succeeding with the first task alone would lead to a theoretically interesting but somewhat academic result since diffusion plays a key role in near-wall (surface) flows where inhomogeneity is at its maximum. This in turn raises two problems: a conceptual one, since at first sight it might seem that spectral methods, such as the ones used in the previous studies (Schiestel 1987; Rubinstein and Zhou 2001), are intrinsically inadequate tools to handle inhomogeneous flows; and a practical one of how to mimic the effects of the diffusion term D_ε without adding it ad hoc to the ε equation.

It is the goal of this paper to show that it is indeed possible to solve both tasks with an extension of the spectral method, the final result being an equation for ε in which the empirical form of the diffusion term D_ε is no longer needed since it is replaced by two terms: the first is ''intrinsic'' in the sense that it is always present as long as the flow is inhomogeneous, and the second term is entirely due to the diffusion of the equation of K. The presence of two diffusion terms in the ε equation explains why, in the surface layer of a planetary boundary layer (PBL), even when the diffusion in the K equation is negligible, the diffusion in the ε equation still plays an important role and cannot be neglected.

2. Extension to inhomogeneous flows

Any extension of a two-point closure formalism for homogeneous flows—defined as those in which random motions have average properties independent of location [Batchelor 1970, see his section 5.3 and Eq. (6.6.1)]—to one in which some features depend on location is necessarily heuristic. We follow the spirit of Saffman (1967), who, while recognizing that ''large-scale structures are unlikely to be isotropic,'' still considered it useful to study the results because of their simplicity. To that consideration, we add an additional one. With a minimal change to the homogeneous formalism, we shall derive results that are physically meaningful and much easier to use than those of the full inhomogeneous, diffusion-based formalism represented by Eq. (1).

Looking ahead, in the case of zero diffusion in the equation for K, the ε equation that we derive has the traditional form:

$$\frac{\partial \varepsilon}{\partial t} = \frac{\varepsilon}{K}(c_1 P - c_2 \varepsilon). \tag{3}$$

But we shall show that $c_{1,2}$ are now functions of both K, ε:

$$c_{1,2} = c_{1,2}(K, \varepsilon). \tag{4}$$

Thus, one can rewrite (3) in the form (1) with constant coefficient terms plus an additional K, ε-dependent term, which will be shown to play the role of D_ε. More specifically, the new term is now a function of the differences

$$D_\varepsilon[c_1(K, \varepsilon) - c_{1,\varepsilon}, c_2(K, \varepsilon) - c_{2,\varepsilon}]. \tag{5}$$

Furthermore, since it will also be shown that for any K, ε

$$\frac{1}{c_1} + \frac{1}{c_2} = \frac{4}{3}, \tag{6}$$

the traditional assumption of *three* independent heuristic variables,

$$D_\varepsilon,\ c_{1,\varepsilon},\ c_{2,\varepsilon} \Rightarrow c_{1,2} = c_{1,2}(K, \varepsilon), \tag{7}$$

is no longer necessary since it is reduced to a *single* function, $c_1 = c_1(K, \varepsilon)$ or $c_2(K, \varepsilon)$, whose form is predicted by the model.

Looking ahead again, in the case of nonzero diffusion D_K in the equation for K, D_ε in (5) will have an additional term representing the effect of D_K.

We shall study the consequences of the new D_ε in three different cases, the first being that of a high-Re, fully developed channel flow, the second that of a planetary boundary layer where turbulence is shear driven and near the ground is highly inhomogeneous, and the third is a comparison of the turbulence integral length scale derived from the present model with the large-eddy simulation (LES) data and the well-known Blackadar relation (Blackadar 1962).

3. Derivation of the ε equation

We follow and generalize previous authors Schiestel (1987) and Rubinstein and Zhou (2001) by considering the following spectra corresponding to a shear-driven flow:

$$E(k) = \begin{cases} Ak^s, & \text{for} \quad k_c \le k \le k_m \\ \mathrm{Ko}\varepsilon^{2/3}k^{-5/3}, & \text{for} \quad k \ge k_m \end{cases}, \tag{8a}$$

$$m(k) = \begin{cases} Bk^s, & \text{for} \quad k_c \le k \le k_m \\ CS\varepsilon^{1/3}k^{-7/3}, & \text{for} \quad k \ge k_m \end{cases}, \tag{8b}$$

where Ko is the Kolmogorov constant, $m(k, t)$ is the Reynolds stress spectrum $-\overline{uw} = \int_{k_c}^{\infty} m(k)\,dk$, $S = \partial U/\partial z$ is the shear, and the production in Eq. (1) is given by $P = -\overline{uw}\,S$ while the form of $m(k)$ is Lumley's spectrum (Pope 2000; Lumley 1967; Canuto and Dubovikov 1996); furthermore, k_m is the wavenumber of the energy containing eddies corresponding to where $E(k)$ and $m(k)$ peak, A and B will be given by Eq. (9), and C is a constant associated with the production term. The generalization with respect to previous studies that is implied in (8a) and (8b) is twofold: first, we generalize the previous authors' IR spectrum as follows:

$$0 \le k \le k_m \ \Rightarrow\ k_c \le k \le k_m, \tag{8c}$$

which at this stage is still an inconsequential mathematical extension. The key departure from previous work is the assumption we shall make in what follows that the lowest wavenumber k_c can be interpreted as

$$k_c^{-1} = \left(\frac{2}{\pi}\right)\min(z, H - z), \tag{8d}$$

where $\min(z, H - z)$ is meant to represent the distance to the nearest boundary and H has to be chosen so as to represent the physical extent of the fluid; Eq. (8d) can also be viewed to describe the largest eddy size allowed by the geometry of the system at any given z. We must stress that both (8c) and (8d) do not affect any of the second-order model derivation that we carry out below and that it is only in the final expressions that (8d) is used (i.e., in the expressions integrated over all wavenumbers); for example the eddy kinetic energy relation,

$$K = \int_{k_c}^{\infty} E(k)\,dk, \tag{8e}$$

is formally identical to the one corresponding to a homogeneous flow in which the lowest wavenumber is zero rather than a finite k_c. The relations (8c) and (8d) also do not affect any second-order closure relation one may employ since in principle the latter are valid at any k and it is only in the final stages that one chooses the value of the lower limit in (8e). One might also consider that the extension to a finite k_c is not done here for the first time. The best example is the case of fully buoyancy-driven flows: it was shown in Canuto and Goldman (1985) that the heat flux constructed using a second-order closure model, once integrated over all wavenumbers beginning at k_c, reproduced the well-known and rather successful mixing length expression (Canuto 2009) with the "mixing length" ℓ identified with the inverse of k_c in the manner suggested in (8d). Going back to relations (8a) and (8b), continuity of the spectra at k_m requires that

$$A = \mathrm{Ko}\varepsilon^{2/3}k_m^{-s-5/3}, \quad B = CS\varepsilon^{1/3}k_m^{-s-7/3}. \tag{9}$$

The turbulent kinetic energy K and the energy production P are easily obtained to be

$$K = \frac{(3s + 5 - 2\psi)}{2(s+1)}\mathrm{Ko}\varepsilon^{2/3}k_m^{-2/3},$$
$$P = \frac{(3s + 7 - 4\psi)}{4(s+1)}CS^2\varepsilon^{1/3}k_m^{-4/3}, \tag{10a}$$

where the variable ψ is defined as follows:

$$\frac{k_c}{k_m} = \psi^{1/(1+s)}. \tag{10b}$$

Clearly, the homogenous limit is recovered by taking $k_c = 0$ and thus $\psi = 0$ in both relations (10a). Taking the time derivatives of the first of (10a) and of (10b), we obtain after some algebra

$$\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial t} = \frac{3}{2}\frac{1}{K}(-D_K + P - \varepsilon) + \left[\frac{1-\psi}{1-2\psi/(3s+5)}\right]\frac{1}{k_m}\frac{\partial k_m}{\partial t}, \tag{11a}$$

where we have used the relation

$$\frac{\partial K}{\partial t} + D_K = P - \varepsilon, \tag{11b}$$

where D_K is the diffusion term in the equation for K. At this point, we can no longer proceed unless we have an additional relation to express the last term in (11a) representing the time derivative of k_m, in terms of the other variables in (11a). As discussed at the beginning, previous authors Schiestel (1987) and Rubinstein and Zhou (2001) assumed the heuristic relation

$$\frac{\partial k_m}{\partial t} = -\beta\frac{\varepsilon}{E(k_m)}, \tag{12}$$

which contains an empirical constant β; also, as noted by Rubinstein and Zhou (2001), the rhs of (12) lacks the production term that should be present by physical considerations. This junction is where we depart from the previous derivations (Schiestel 1987; Rubinstein and Zhou 2001) and substitute the ad hoc relation (12) with one based on a more solid foundation. To do so, we substitute the spectra (8a) and (8b) into the well-known dynamic equation governing the spectrum $E(k, t)$ (Batchelor 1970):

$$\frac{\partial E(k,t)}{\partial t} = -\underbrace{\mathcal{D}_K(k,t)}_{\text{diffusion}} + \underbrace{m(k,t)S}_{\text{source}} + \underbrace{T(k,t)}_{\text{transfer}} - \underbrace{2\nu k^2 E(k,t)}_{\text{dissipation}}, \tag{13}$$

where $\mathcal{D}_K(k, t)$ is the diffusion in wavenumber space, $T(k, t)$ is the transfer function due to the nonlinear interactions, and the last term represents the dissipation by the kinematic viscosity ν. Since the key problem of turbulence is the modeling of $T(k)$, the use of (13) is predicated on knowing such a function. In this derivation, we need to apply (13) only to the infrared region $k_c \le k < k_m$, where we prove that

$$-\mathcal{D}_K(k, t) + T(k, t) = 0, \quad k_c \le k < k_m. \tag{14}$$

To that end, for $k_c \le k < k_m$ we substitute (8a) and (8b) into (13) and neglect the viscous term that does not affect the large-eddy infrared region. The result is

$$\frac{\partial A}{\partial t} = BS + [-\mathcal{D}_K(k) + T(k)]k^{-s}, \quad k_c \le k < k_m. \tag{15}$$

Since both A and B are independent of k, one must have

$$-\mathcal{D}_K(k) + T(k) = ck^s, \quad k_c \le k < k_m, \tag{16}$$

where c is a constant. Assuming, as is usually done, that turbulence transfers no energy to or from the scale k_c^{-1}, this means that at the specific wavenumber k_c, we have $-\mathcal{D}_K(k_c) + T(k_c) = 0$. Inserting this result in Eq. (16), it follows that c must be zero, which thus proves (14). Strictly speaking, this is not an exact result since if we had used smooth spectra rather than the form in (8a), $-\mathcal{D}_K(k) + T(k)$, while still zero at $k = k_c$, would have begun to be nonzero near k_m; this means that in adopting (14) we are approximating a function smoothly rising from zero near $k = k_m$ with a Heaviside-like 0–1 function. Because of the smallness of the region near k_m where $-\mathcal{D}_K(k) + T(k)$ is not zero, we believe that (14) is not a bad approximation. In any case, (14) is not an additional assumption; it is the result of the starting split-spectrum (8a). Using (8a) and (8b), the second part of (10a) and (14) in (13), and neglecting the last term for the interval $k_c < k < k_m$, we obtain the new relation that substitutes the heuristic relation (12):

$$\frac{3s+5}{2}\frac{1}{k_m}\frac{\partial k_m}{\partial t} = \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial t} - \frac{3(3s+5-2\psi)}{(3s+7-4\psi)}\frac{P}{K}. \tag{17}$$

Solving (11a) and (17) for the two variables $\partial k_m/\partial t$ and $\partial \varepsilon/\partial t$, the latter has the following form:

$$\frac{\partial \varepsilon}{\partial t} = \frac{\varepsilon}{K}(-c_2 D_K + c_1 P - c_2 \varepsilon), \tag{18a}$$

where

$$c_1 = \frac{3}{2} - \frac{3(1-\psi)}{3(s+1)+4(1-\psi)}, \quad c_2 = \frac{3}{2} + \frac{1-\psi}{s+1}; \tag{18b}$$

thus, for any ψ we have the relation

$$\frac{1}{c_1} + \frac{1}{c_2} = \frac{4}{3}. \tag{18c}$$

Equation (18c) shows that the harmonic mean of c_1 and c_2 is 3/2. To complete the derivation, we need to relate internal variable ψ to the basic variables K and ε. This is done by combining the first part of (10a) with (10b). The result is the following algebraic relation expressing ψ in terms of the dynamical variables K and ε:

$$\left(1+\frac{2}{3}\frac{1-\psi}{s+1}\right)\psi^p = \frac{K}{K_*}, \quad p \equiv \frac{2}{3(1+s)},$$
$$K_* \equiv \frac{3}{2}\mathrm{Ko}\varepsilon^{2/3}k_c^{-2/3}, \tag{18d}$$

where at $\psi = 1$, $K = K_*$. In addition, in terms of ψ, the boundary condition implied by (8c), (8d), and (10b) is as follows:

$$\psi \leq 1. \tag{18e}$$

Thus, Eqs. (18a)–(18e) represent the complete form of the new ε equation.

4. Some implications of the new ε equations [Eqs. (18a) and (18b)]

From Eqs. (11a), (17), (10b), and (18e) we also obtain

$$\psi < 1: \quad \frac{\partial k_m}{\partial t} = -\frac{2}{3s+5}\frac{k_m}{K}(c_2 D_K + c_1 P + c_2\varepsilon) \tag{19a}$$

and

$$\psi = 1: \quad k_m = k_c. \tag{19b}$$

Equations (19a) and (19b) show that k_m decreases with time until it reaches its minimum value k_c (i.e., $\psi = 1$), where the governing Eq. (19a) yields to the boundary condition (19b). Another way of expressing the same result is by saying that the energy-containing eddies grow in time until they reach their maximum size (twice of) z or $H - z$. Thus, as time evolves, the energy-containing eddies reach the boundary where the flow is highly inhomogeneous. By the same argument, the regime $\psi \ll 1$ is only found away from the boundaries where the flow is more homogeneous.

Now, consider the homogeneous case characterized by $\psi \equiv (k_c/k_m)^{s+1} \ll 1$. Equation (18b) gives

$$\psi \ll 1:$$

$$c_1 \to c_{1,\varepsilon} = \frac{3(3s+5)}{2(3s+7)} = \begin{cases} \simeq 1.27, & \text{for} \quad s=2 \\ \simeq 1.34, & \text{for} \quad s=4 \end{cases}, \tag{20a}$$

$$c_2 \to c_{2,\varepsilon} = \frac{3s+5}{2(s+1)} = \begin{cases} \simeq 1.83, & \text{for} \quad s=2 \\ = 1.70, & \text{for} \quad s=4 \end{cases}. \tag{20b}$$

The results show that in the homogeneous case, $c_{1,2}$ are indeed constant as in Eq. (1) where the inhomogeneity represented by the diffusion term D_ε is now zero. We may also note that the values of the constants predicted by the model are close to the compromise values cited earlier. On the other hand, in the inhomogeneous case, when the eddy size approaches its maximum possible value, $\psi \equiv (k_c/k_m)^{s+1} \to 1$, Eq. (18b) yields

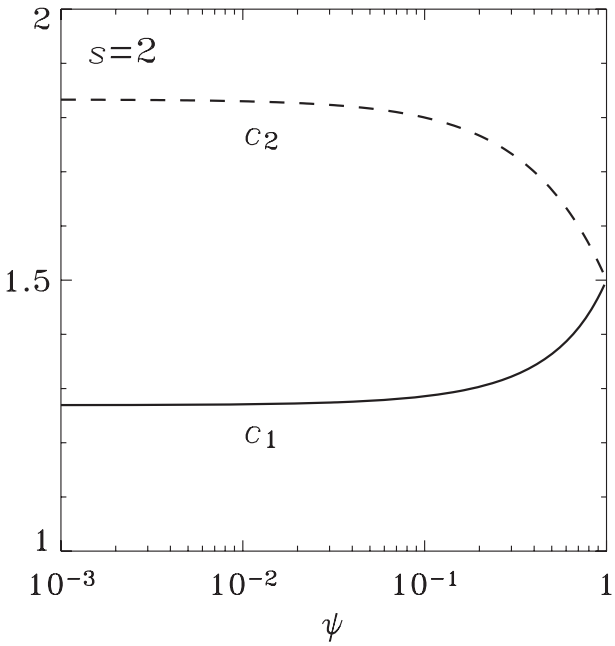

FIG. 1. The c_1 (solid line) and c_2 (dashed line) obtained from the new split-spectrum model vs $\psi = (k_c/k_m)^{s+1}$ according to (18b) for $s = 2$. For $\psi \ll 1$, c_1 and c_2 are constants and $c_1 < c_2$; as $\psi \to 1$, both c_1 and c_2 approach the value 1.5, their harmonic mean.

$$\psi = 1: \quad c_1 = c_2 = \frac{3}{2}, \tag{21}$$

which gives a physical interpretation to the harmonic mean discussed after Eq. (18c), which is now seen to correspond to the inhomogeneous limit $k_c/k_m \to 1$.

Next, we show that $\psi = 1$ leads to the steady state. Using the second part of (10a), (11b), (18a), (19a) and (19b), we derive the relation

$$\psi = 1: \quad \frac{\partial}{\partial t}\left(\frac{P}{\varepsilon}\right) = -\frac{P}{\varepsilon K}\frac{\partial K}{\partial t}, \tag{22}$$

which implies that if $\partial K/\partial t > 0$, then P/ε will decrease toward $\partial K/\partial t = 0$; if $\partial K/\partial t < 0$, then P/ε will increase toward $\partial K/\partial t = 0$. Thus, at $\psi = 1$, if the flow is not in a steady state, it will relax to it.

In Fig. 1, we plot c_1 and c_2 versus ψ for $s = 2$. For $\psi \ll 1$, away from the steady-state limit, c_1 and c_2 are indeed constant with values close to the empirical values; on the other hand, as ψ increases toward unity, c_1 and c_2 exhibit opposite behaviors (namely, c_1 increases while c_2 decreases, reaching the common value $3/2$ at $\psi = 1$) and thus it becomes possible for the flow to approach the

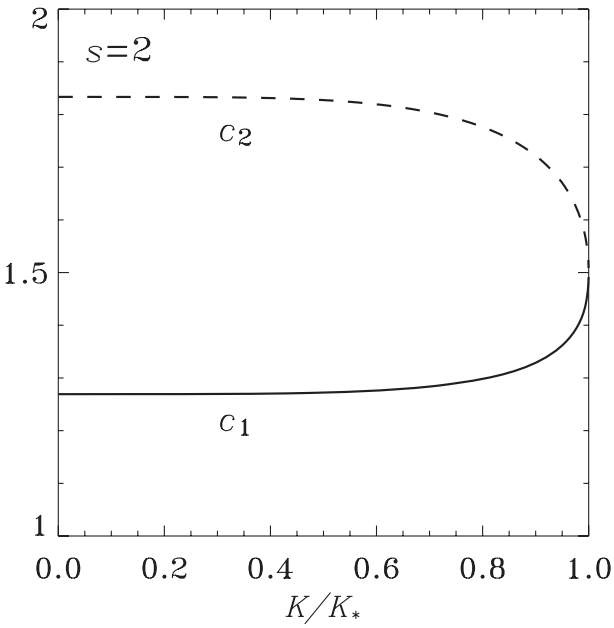

FIG. 2. As in Fig. 1, but vs K/K_* according to (18b) and (18d). For $K/K_* \ll 1$, c_1 and c_2 are constants and $c_1 < c_2$; as $K/K_* \to 1$, both c_1 and c_2 approach the value 1.5.

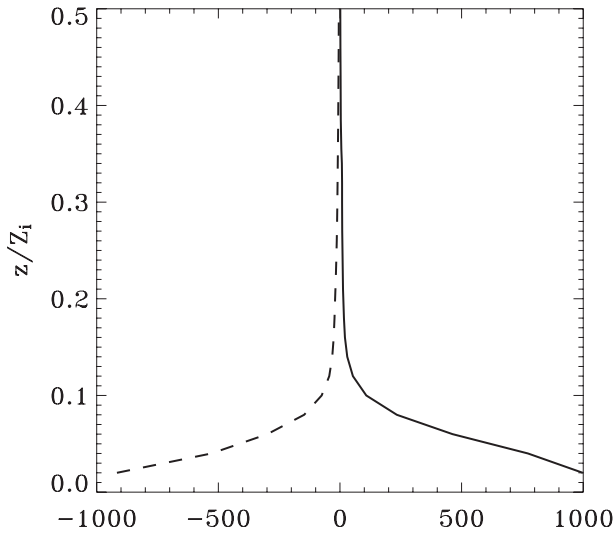

FIG. 3. For a shear-driven PBL of height Z_i and friction velocity u_*, the solid line represents the diffusion-like term $-D_\varepsilon$, given by the present model Eq. (28) (normalized by $u_*^4 Z_i^{-2}$), in the ε equation [Eq. (1)]. The LES data to compute Eq. (28) are taken from Moeng and Sullivan (1994). The dashed line represents $\varepsilon(c_{1,\varepsilon}P - c_{2,\varepsilon}\varepsilon)/K$ (normalized by $u_*^4 Z_i^{-2}$) in Eq. (1). The figure shows that near the boundary, where the flow is maximally inhomogeneous, the two terms cancel out and thus a steady state is possible, consistent with observations.

steady state. In Fig. 2, we used (18b) and (18d) to plot c_1 and c_2 versus K/K_* for the $s = 2$ case.

5. Tests of the new model

Here, we consider three tests of the new ε equation. We begin with a high-Re, fully developed channel flow as discussed in Pope [2000, see his Eqs. (10.69)–(71)]. In the log-wall region production and dissipation balance, the diffusion term in the K equation vanishes, leading to a constant K. The stationary limit of Eq. (1) then becomes

$$0 = -D_\varepsilon + \frac{\varepsilon^2}{K}(c_{1,\varepsilon} - c_{2,\varepsilon}). \tag{23}$$

Using the down-gradient closure,

$$D_\varepsilon = -\frac{\partial}{\partial y}\left(\nu_t \frac{\partial \varepsilon}{\partial y}\right), \quad \nu_t = C_\mu \frac{K^2}{\varepsilon}, \tag{24}$$

where y is the distance from the boundary. The solution of Eq. (23) then reads

$$\varepsilon = A\frac{K^{3/2}}{y}, \quad A^2 = \frac{C_\mu}{(c_{2,\varepsilon} - c_{1,\varepsilon})}. \tag{25}$$

In the present model [(18a) and (18b)], the above conditions lead to the following relations:

$$c_1 = c_2 \Rightarrow \psi = 1 \Rightarrow K \Rightarrow K_*. \tag{26}$$

Using the third part of (18d) and (8d), we obtain

$$\varepsilon = B\frac{K^{3/2}}{y}, \quad B = \frac{\pi}{2}\left(\frac{2}{3\mathrm{Ko}}\right)^{3/2}. \tag{27}$$

The first part of (27) reproduces the first part of (25). Using the standard values $C_\mu = 0.09$, Ko $= 1.6$, $c_{1,\varepsilon} = 1.44$, and $c_{2,\varepsilon} = 1.92$, the two coefficients $A = 0.43$ and $B = 0.42$ are also very close to each other.

In the second test, we employ large-eddy simulation data (Moeng and Sullivan 1994) for a shear-driven PBL to compute the variables in (5). In Fig. 3 we plot the $-D_\varepsilon$ term in Eq. (1) (solid line) given by Eq. (28) below versus z/Z_i, the height from the ground $z = 0$ normalized by the PBL depth Z_i. The explicit form of D_ε is now found to be

$$D_\varepsilon = \frac{\varepsilon}{K}[c_2 D_K - (c_1 - c_{1,\varepsilon})P + (c_2 - c_{2,\varepsilon})\varepsilon]. \tag{28}$$

Using the same LES data, we also plot (dashed line) the $\varepsilon(c_{1,\varepsilon}P - c_{2,\varepsilon}\varepsilon)/K$ term in Eq. (1). As can be observed, near the boundary where z/Z_i is small and the flow is highly inhomogeneous, the two terms cancel out and a steady state is therefore possible. This means that (28)

plays the role of the diffusion term in (1), with the difference that in the present formalism there is no longer the necessity to introduce the heuristic diffusion term D_ε as in the standard treatment (Umlauf and Burchard 2005) since it is now given by Eq. (28).

In the third test, which also deals with the PBL, we show how to recover the LES data and Blackadar's formula for the dissipation length scale in the small z limit. The first part of Eq. (10a) can be rewritten in terms of the integral length scale L_ε:

$$\varepsilon = \frac{(2K)^{3/2}}{L_\varepsilon}, \tag{29}$$

with

$$L_\varepsilon = (3\mathrm{Ko})^{3/2}\left(1 + \frac{2}{3}\frac{1-\psi}{s+1}\right)^{3/2}\psi^{1/(s+1)}k_c^{-1}. \tag{30}$$

In the PBL case, the variable H in Eq. (8d) can be regarded as infinity since the geometry is half open, $k_c^{-1} = 2z/\pi$, and thus Eq. (30) becomes

$$L_\varepsilon = \frac{2(3\mathrm{Ko})^{3/2}}{\pi}\left(1 + \frac{2}{3}\frac{1-\psi}{s+1}\right)^{3/2}\psi^{1/(s+1)}z. \tag{31}$$

Near the surface, for small z, $\psi \to 1$, Eq. (31) gives

$$z \to 0: \quad L_\varepsilon = \frac{2(3\mathrm{Ko})^{3/2}}{\pi}z = 16.7\kappa z, \tag{32}$$

where $\kappa = 0.4$ is the Von Kármán constant, and Ko = 1.6. Rewriting (32) in terms of the commonly used length scale formula ℓ as

$$L_\varepsilon = B_1\ell, \quad B_1 = (2K_s)^{3/2}/u_*^3 \approx 16.6, \tag{33a}$$

where (33a) results from equating the production and dissipation rates in the log-law region. Here K_s is the value of K at the surface and u_* is the friction velocity. The ratio $K_s^{1/2}/u_*$ determined from the average of eight different measurements (Mellor and Yamada 1982) yields $B_1 \approx 16.6$, as in (33a). The constant 16.7 in (32) is very close to the measured value of B_1 in (33a). In addition, from (31) we obtain that

$$\ell \simeq \kappa z, \quad \text{as} \quad z \to 0. \tag{33b}$$

Thus, the present model [(31)] recovers the well-known small z expression.

Away from the surface, inhomogeneity is reduced and ψ becomes less than one, and thus the growth of L_ε given by (31) is less than linear in z. This behavior is in agreement with Blackadar's length scale formula. The full z dependence of L_ε requires the solution of the complete $K - \varepsilon$ model.

6. Conclusions

We have used a two-point closure spectrum to derive a new ε equation, Eqs. (18). With respect to previous attempts (Schiestel 1987; Rubinstein and Zhou 2001) using a similar methodology, we substituted the empirical closure (12) suggested in Schiestel (1987) with a new relation based on a physical model of the transfer function $T(k, t)$ in the infrared region. In the homogeneous case, the ε equation contains only production and dissipation terms with constant coefficients and zero diffusion D_ε. The present model predicts such a form with the values of the constants $c_{1,2}$ close to the "compromise" values used in the literature. We note that the procedure we suggest has no adjustable parameters.

To encompass inhomogeneous flows, we have made a simple suggestion: traditionally, the diffusion term D_ε [see Eq. (1)] requires a closure whereas in the new formalism such a term is replaced by relation (28), which is given by the model without the need to invoke additional closure relations. A part of the new diffusion-like term is due to the nonzero difference between $c_1(K, \varepsilon)$ and $c_2(K, \varepsilon)$, which are now functions of both K, ε, with their constant counterparts $c_{1,\varepsilon}$, $c_{2,\varepsilon}$ of the homogeneous case; the other part of the diffusion in the ε equation is simply inherited from the diffusion in the K equation. The three heuristic variables (7) of the traditional ε equation [Eq. (1)] are now replaced by one function (c_1 or c_2) plus an additional one related to the diffusion in the K equation.

To allow realistic applications to geophysical flows, buoyancy effects of both stable and unstable stratification must be included in the derivation of the ε equation. Work is in progress to extend the above formalism to the case of a shear- and buoyancy-driven flow. The present study lays a foundation for the new approach.

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