Mode decomposition scheme for ideal magnetohydrodynamic plane waves in space-time coordinates

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[1] Exact nonsinusoidal plane wave solutions of the linearized equations of ideal magnetohydrodynamics are used to develop a decomposition scheme for extracting the forward and backward propagating components of the fast, slow, and Alfvén modes from measured data. The decomposition technique is formulated in the space-time domain for waves propagating in one direction. The different wave modes are extracted by means of projection operators that are expressed in matrix form. Because the elements of these matrices are constants, the same projection operators can be used to obtain the mode decomposition in the frequency (Fourier) domain. The projection operators are identical to those obtained by Glassmeier et al. (1995) although they are derived here by different means. In the case of wave propagation parallel or perpendicular to the background magnetic field the wave modes are degenerate and require separate treatment. These special cases are included in the present analysis so the resulting decomposition scheme encompasses all possible propagation directions.

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1. Introduction

[2] The purpose of this paper is to show that the decomposition scheme for linear magnetohydrodynamic (MHD) wavefields developed by *Glassmeier et al.* [1995] can be formulated in the space-time domain rather than the frequency-wave number domain (Fourier domain). The decomposition of MHD wavefields into forward propagating and backward propagating fast, slow, and Alfvén wave components can be regarded as an eigenvalue problem in the six-dimensional configuration space. The mode decomposition is achieved by projecting the state vector onto each of six linearly independent eigenvectors. Thus the method is completely specified once these six projection operators are defined.

[3] The advantage of the method developed here is that the decomposition can be performed in real time using the measured variables $\boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{b}(\boldsymbol{x}, t)$, and $\rho(\boldsymbol{x}, t)$ so there is no need to Fourier transform the data. Here, $\boldsymbol{v}(\boldsymbol{x}, t)$ is the fluid velocity, $\boldsymbol{b}(\boldsymbol{x}, t)$ is the magnetic field vector, and $\rho(\boldsymbol{x}, t)$ is the mass density. The primary disadvantage of the method is that it only applies to wavefields having one direction of propagation. Special attention is given to the degenerate cases that occur when the propagation direction is either parallel ($\theta = 0$) or perpendicular ($|\theta| = \pi/2$) to the background magnetic field $\boldsymbol{B}_0$. These special cases require a different decomposition scheme than the one developed by *Glassmeier et al.* [1995]. These results, therefore, are new. To provide a complete decomposition algorithm, and because they are needed in some applications, all angles of propagation in the range $|\theta| < \pi$ are included in the present treatment.

[4] The outline of this paper is as follows. The analysis of waves propagating parallel to $\boldsymbol{B}_0$ is presented section 2. The case of waves propagating at an arbitrary direction to $\boldsymbol{B}_0$ is presented section 3. The analysis of waves propagating perpendicular to $\boldsymbol{B}_0$ is presented section 4. The conclusions are summarized in section 5.

2. Waves Propagating Parallel to $\boldsymbol{B}_0$

[5] Choose a system of orthogonal Cartesian coordinates (x, y, z) such that the background magnetic field $\boldsymbol{B}_0$ is aligned with the x-direction. Thus $\boldsymbol{B}_0 = B_0\boldsymbol{e}_x$ where $B_0 > 0$ and $\boldsymbol{e}_x$ is the unit vector in the x-direction. For plane waves that propagate in the x-direction $\partial/\partial y = \partial/\partial z = 0$ and the linearized equations of ideal MHD take the form

$$\frac{\partial v_x}{\partial t} = -\frac{c_s^2}{\rho_0}\frac{\partial \rho}{\partial x}, \tag{1}$$

$$\frac{\partial v_y}{\partial t} = V_A\frac{\partial b_y}{\partial x}, \tag{2}$$

$$\frac{\partial v_z}{\partial t} = V_A\frac{\partial b_z}{\partial x}, \tag{3}$$

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$$\frac{\partial b_y}{\partial t} = V_A \frac{\partial v_y}{\partial x}, \tag{4}$$

$$\frac{\partial b_z}{\partial t} = V_A \frac{\partial v_z}{\partial x} \tag{5}$$

$$\frac{\partial \rho}{\partial t} = -\rho_0 \frac{\partial v_x}{\partial x}, \tag{6}$$

where $\boldsymbol{b}$ is expressed in velocity units by dividing by $(\rho_0\,\mu_0)^{1/2}$, $V_A = B_0/(\rho_0\,\mu_0)^{1/2}$ is the Alfvén speed, c_s is the sound speed, μ_0 is the permeability of free space, and $b_x = 0$. The parameters B_0, V_A, c_s, and ρ_0 which characterize the ambient medium are all constant. The condition $\nabla \cdot \boldsymbol{b} = 0$ is identically satisfied. The polytropic equation of state $p/\rho^\gamma = \text{const}$, where γ is the ratio of specific heats, has been used to rewrite the pressure gradient term in equation (1). Note that the equations for v_x and ρ are decoupled from the other equations; these are equations for sound waves propagating parallel to $\boldsymbol{B}_0$. The remaining four equations describe Alfvén waves.

2.1. Alfvén Waves

[6] Equations (2), (3), (4), and (5) describe Alfvén waves propagating along the background magnetic field $\boldsymbol{B}_0$. The wavefields $\boldsymbol{v} = (v_y, v_z)$ and $\boldsymbol{b} = (b_y, b_z)$ are both polarized perpendicular to $\boldsymbol{B}_0$. The general solution of the Alfvén wave equations can be written

$$\boldsymbol{v}(x,t) = +\boldsymbol{f}\left(t - \frac{x}{V_A}\right) + \boldsymbol{g}\left(t + \frac{x}{V_A}\right), \tag{7}$$

$$\boldsymbol{b}(x,t) = -\boldsymbol{f}\left(t - \frac{x}{V_A}\right) + \boldsymbol{g}\left(t + \frac{x}{V_A}\right), \tag{8}$$

where $\boldsymbol{f} = (f_1, f_2)$, $\boldsymbol{g} = (g_1, g_2)$, and f_1, f_2, g_1, and g_2 are four arbitrary functions. From inspection, it is obvious that the forward ($+x$) propagating wave components are given by

$$\boldsymbol{v}_+(x,t) = +\boldsymbol{f}\left(t - \frac{x}{V_A}\right), \tag{9}$$

$$\boldsymbol{b}_+(x,t) = -\boldsymbol{f}\left(t - \frac{x}{V_A}\right), \tag{10}$$

and the backward ($-x$) propagating wave components are given by

$$\boldsymbol{v}_-(x,t) = \boldsymbol{g}\left(t + \frac{x}{V_A}\right), \tag{11}$$

$$\boldsymbol{b}_-(x,t) = \boldsymbol{g}\left(t + \frac{x}{V_A}\right). \tag{12}$$

The solutions satisfy the well-known relation $\boldsymbol{b}_+ = -\boldsymbol{v}_+$ for forward propagating Alfvén waves and $\boldsymbol{b}_- = +\boldsymbol{v}_-$ for backward propagating Alfvén waves.

[7] If the wavefields $\boldsymbol{v}$ and $\boldsymbol{b}$ are both measured at $x = 0$, then the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ can be derived from the relations

$$\boldsymbol{f}(t) = \frac{1}{2}[\boldsymbol{v}(0,t) - \boldsymbol{b}(0,t)], \tag{13}$$

$$\boldsymbol{g}(t) = \frac{1}{2}[\boldsymbol{v}(0,t) + \boldsymbol{b}(0,t)]. \tag{14}$$

The complete solutions can then be constructed for all x and t from the formulas (7) and (8).

[8] If $\boldsymbol{v}$ and $\boldsymbol{b}$ are both measured at the point x as a function of time t, then the forward propagating components of the wavefields can be determined from the relations

$$\begin{pmatrix} \boldsymbol{v}_+ \\ \boldsymbol{b}_+ \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{b} \end{pmatrix} \tag{15}$$

or

$$\begin{pmatrix} \boldsymbol{v}_+ \\ \boldsymbol{b}_+ \end{pmatrix} = P^+\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{b} \end{pmatrix} \tag{16}$$

where

$$P^+ = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{17}$$

is the projection operator. The backward propagating components of the wavefields can be determined from the relations

$$\begin{pmatrix} v_- \\ b_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v \\ b \end{pmatrix} \tag{18}$$

or

$$\begin{pmatrix} v_- \\ b_- \end{pmatrix} = P^-\begin{pmatrix} v \\ b \end{pmatrix} \tag{19}$$

where

$$P^- = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{20}$$

is the projection operator. These relations provide a real time projection method for the decomposition of the Alfvén wave fields into forward and backward travelling components. In all of the above expressions the wavefields are evaluated at (x, t) so that $\boldsymbol{v} = \boldsymbol{v}(x, t)$, $\boldsymbol{b} = \boldsymbol{b}(x, t)$, and so on.

[9] Analysis of the waves in the frequency domain are achieved by means of the Fourier transform

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{i\omega t}\, dt. \tag{21}$$

For example, the Fourier transform of the forward wave component $\boldsymbol{v}_+(x, t)$ yields

$$\boldsymbol{V}_+(x,\omega) = \boldsymbol{F}(\omega) e^{i\omega x/V_A}. \tag{22}$$

The Fourier transform with respect to the spatial variable x then yields

$$\boldsymbol{V}_+(k,\omega) = 2\pi \boldsymbol{F}(\omega)\delta(k - \omega/V_A), \tag{23}$$

where $\delta(k)$ is the Dirac delta function. Thus the wave number spectrum is related to the frequency spectrum $\boldsymbol{F}(\omega)$ by the substitution $\omega = kV_A$. The backward propagating wave component is

$$\boldsymbol{V}_-(k,\omega) = 2\pi \boldsymbol{G}(\omega)\delta(k + \omega/V_A) \tag{24}$$

so that the wave number spectrum is related to the frequency spectrum $\boldsymbol{G}(\omega)$ by the substitution $\omega = -kV_A$. The dispersion relations for the waves is enforced by the delta function in wave number-frequency space.

[10] It is now shown that the decomposition into forward and backward propagating wave components takes the same form in the Fourier domain as in the space-time domain. Suppose the fields $\boldsymbol{v}$ and $\boldsymbol{b}$ are measured as functions of time at the point $x = 0$ and that the Fourier transform of the measured fields is then computed. The Fourier transform of equations (7) and (8) shows that

$$\boldsymbol{V}(0,\omega) = +\boldsymbol{F}(\omega) + \boldsymbol{G}(\omega), \tag{25}$$

$$\boldsymbol{B}(0,\omega) = -\boldsymbol{F}(\omega) + \boldsymbol{G}(\omega). \tag{26}$$

This can be inverted to obtain

$$\boldsymbol{F}(\omega) = \frac{1}{2}[\boldsymbol{V}(0,\omega) - \boldsymbol{B}(0,\omega)], \tag{27}$$

$$\boldsymbol{G}(\omega) = \frac{1}{2}[\boldsymbol{V}(0,\omega) + \boldsymbol{B}(0,\omega)]. \tag{28}$$

Likewise, the decomposition into forward and backward propagating components are given by the Fourier transforms (with respect to time) of equations (16) and (19) evaluated at $x = 0$. This yields

$$\begin{pmatrix} \boldsymbol{V}_+ \\ \boldsymbol{B}_+ \end{pmatrix} = P^+ \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{B} \end{pmatrix} \tag{29}$$

and

$$\begin{pmatrix} \boldsymbol{V}_- \\ \boldsymbol{B}_- \end{pmatrix} = P^- \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{B} \end{pmatrix} \tag{30}$$

where the projection matrices are given by equations (17) and (20). Hence the same projection method works in both the space-time domain and the frequency domain. The formulations in the two domains are, therefore, equivalent. The advantages of operating in the space-time domain instead of the frequency domain are that the decomposition can be performed in real time and there is no need to Fourier transform the measured signals prior to the mode decomposition procedure. However, the method can only be applied when the wave propagation is unidirectional.

[11] The mode decomposition framework can also be applied to random signals (stochastic processes). In the linear theory of wave propagation there is no mode coupling and, therefore, it is logical to assume that forward and backward propagating components f_1, g_1, f_2, and g_2 are mutually stochastically independent or otherwise uncorrelated processes. As a consequence, the power spectrum of the stationary process $v_y(x, t)$ is equal to the sum of the power spectra of f_1 and g_1. The power spectra of the other wave components are computed in the same way.

2.2. Sound Waves

[12] For plane wave propagation parallel to $\boldsymbol{B}_0$ there exists only one acoustic mode that propagates at the sound speed c_s. The wave normal diagram (Friedrichs diagram) shows that this is the slow MHD mode when $V_A > c_s$ and the fast MHD mode when $V_A < c_s$. The general solution of the governing equations (1) and (6) takes the form

$$v_x(x,t) = f\left(t - \frac{x}{c_s}\right) - g\left(t + \frac{x}{c_s}\right), \tag{31}$$

$$\frac{c_s\rho(x,t)}{\rho_0} = f\left(t - \frac{x}{c_s}\right) + g\left(t + \frac{x}{c_s}\right), \tag{32}$$

where f and g are arbitrary functions. By inspection, the forward propagating wave components are given by

$$v_x^+(x,t) = f\left(t - \frac{x}{c_s}\right), \tag{33}$$

$$\frac{c_s\rho^+(x,t)}{\rho_0} = f\left(t - \frac{x}{c_s}\right), \tag{34}$$

and the backward propagating wave components are given by

$$v_x^-(x,t) = -g\left(t + \frac{x}{c_s}\right), \tag{35}$$

$$\frac{c_s\rho^-(x,t)}{\rho_0} = +g\left(t + \frac{x}{c_s}\right). \tag{36}$$

This observation is the key to the decomposition procedure.

[13] If the wavefields v_x and ρ are both measured at $x = 0$, then the functions f and g can be derived from the relations

$$f(t) = \frac{1}{2}\left[+v_x(0,t) + \frac{c_s\rho(0,t)}{\rho_0}\right], \tag{37}$$

$$g(t) = \frac{1}{2}\left[-v_x(0,t) + \frac{c_s\rho(0,t)}{\rho_0}\right]. \tag{38}$$

The complete solutions can then be constructed for all x and t from the formulas (31) and (32).

[14] If v_x and ρ are measured at the point x as a function of time t, then the forward propagating components of the wave fields can be determined from the relation

$$\boldsymbol{v}^+ = P^+\boldsymbol{v} \tag{39}$$

where

$$\boldsymbol{v} = \begin{pmatrix} v_x \\ c_s\rho/\rho_0 \end{pmatrix} \tag{40}$$

and

$$P^+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{41}$$

is the projection operator. The backward propagating components of the wavefields can be determined from the relation

$$\boldsymbol{v}^- = P^-\boldsymbol{v} \tag{42}$$

where

$$P^- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{43}$$

These projection operators provide a real-time decomposition of the acoustic wavefields into forward and backward travelling components.

[15] As shown for the Alfvén wave modes in the previous subsection, the mode decomposition procedure in the Fourier domain takes the same form as in the space-time domain. One can obtain the necessary relations by setting $x = 0$ and then taking the Fourier transform of all the equations in this subsection. The generalization to random fields is also performed as described in the previous subsection.

3. Waves Propagating at an Arbitrary Angle to $\boldsymbol{B_0}$

[16] Choose a coordinate system so that the background magnetic field $\boldsymbol{B}_0$ lies in the xz-plane and makes an angle θ with the x-axis. Thus $\boldsymbol{B}_0 = B_0(\cos\theta, 0, \sin\theta)$, where $B_0 > 0$ and $|\theta| \neq 0, \pi, \pi/2$. For plane wave propagation in the $\pm x$-direction $\partial/\partial y = \partial/\partial z = 0$ and the linearized equations of ideal MHD become

$$\frac{\partial v_x}{\partial t} = -\frac{c_s^2}{\rho_0}\frac{\partial \rho}{\partial x} - V_A \sin\theta \frac{\partial b_z}{\partial x}, \tag{44}$$

$$\frac{\partial v_y}{\partial t} = \quad 0 \quad + V_A \cos\theta \frac{\partial b_y}{\partial x}, \tag{45}$$

$$\frac{\partial v_z}{\partial t} = \quad 0 \quad + V_A \cos\theta \frac{\partial b_z}{\partial x}, \tag{46}$$

$$\frac{\partial b_y}{\partial t} = V_A \cos\theta \frac{\partial v_y}{\partial x}, \tag{47}$$

$$\frac{\partial b_z}{\partial t} = V_A \frac{\partial}{\partial x}(v_z \cos\theta - v_x \sin\theta), \tag{48}$$

$$\frac{\partial \rho}{\partial t} = -\rho_0 \frac{\partial v_x}{\partial x}, \tag{49}$$

where $\boldsymbol{b}$ is expressed in Alfvén units and $b_x = 0$. The condition $\nabla \cdot \boldsymbol{b} = 0$ is also satisfied.

3.1. Alfvén Waves

[17] The equations for the y-components (45) and (47) are decoupled from the other components and from the equations for the Alfvén waves. The general solution for the Alfvén wave components takes the form

$$v_y(x,t) = +f\left(t - \frac{x}{V_A\cos\theta}\right) + g\left(t + \frac{x}{V_A\cos\theta}\right), \tag{50}$$

$$b_y(x,t) = -f\left(t - \frac{x}{V_A\cos\theta}\right) + g\left(t + \frac{x}{V_A\cos\theta}\right), \tag{51}$$

where f and g are two arbitrary functions. The phase speed of the waves f and g is $v_{ph} = \pm V_A \cos\theta$, respectively. By inspection of the general solution, the forward propagating Alfvén wave component is obtained by setting $g = 0$ so that

$$\begin{pmatrix} v_y^+ \\ b_y^+ \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} v_y \\ b_y \end{pmatrix} \equiv P^+\begin{pmatrix} v_y \\ b_y \end{pmatrix}, \tag{52}$$

where P^+ is the projection operator. The backward propagating Alfvén wave component is obtained by setting $f = 0$ so that

$$\begin{pmatrix} v_y^- \\ b_y^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_y \\ b_y \end{pmatrix} \equiv P^-\begin{pmatrix} v_y \\ b_y \end{pmatrix}. \tag{53}$$

Here the dependent variables are evaluated at the point (x, t). This is the desired decomposition into forward and backward propagating components.

[18] In terms of the unit vectors

$$e_+^A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad e_-^A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{54}$$

the projection operators may be written

$$P^+ = e_+ e_+^T, \qquad P^- = e_- e_-^T. \tag{55}$$

where the 'T' indicates the vector transpose. These are equivalent to the projection operators in the work of *Glassmeier et al.* [1995] except that in the derivation presented here these operators operate directly on space-time functions rather than functions in the Fourier domain.

[19] The above results may also be expressed in the frequency domain. From equations (50) and (51) the Fourier transform of $v_y(x = 0, t)$ and $b_y(x = 0, t)$ are

$$V_y(x = 0, \omega) = +F(\omega) + G(\omega), \tag{56}$$

$$B_y(x = 0, \omega) = -F(\omega) + G(\omega), \tag{57}$$

where $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$. Therefore if the frequency spectra $V_y(x = 0, \omega)$ and $B_y(x = 0, \omega)$ are measured, then the spectra of the forward and backward propagating components $F(\omega)$ and $G(\omega)$ can be determined. In fact, the Fourier transformed quantities obey the same projection relations (52) and (53) as the space-time variables.

[20] In the case of random (stochastic) wavefields it is consistent with linear wave theory to assume that the forward and backward propagating signals are independent processes (or uncorrelated). Thus the power spectrum of $f(t) + g(t)$ is equal to the sum of the power spectra of f and g.

3.2. Fast and Slow Waves

[21] The equations for the fast and slow waves (44), (46), (48), and (49) form a linear system with constant coefficients that can be written

$$\frac{\partial \boldsymbol{v}}{\partial t} + A\frac{\partial \boldsymbol{v}}{\partial x} = 0, \tag{58}$$

where $\boldsymbol{v} = (v_x, v_z, b_z, c_s\rho/\rho_0)^T$ and

$$A = \begin{bmatrix} 0 & 0 & V_A \sin\theta & c_s \\ 0 & 0 & -V_A\cos\theta & 0 \\ V_A\sin\theta & -V_A\cos\theta & 0 & 0 \\ c_s & 0 & 0 & 0 \end{bmatrix}. \tag{59}$$

This linear system is solved using standard techniques, and without the use of Fourier transforms, in the following way.

[22] The eigenvalues of the matrix A are written

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(+v_f, -v_f, +v_s, -v_s\right), \tag{60}$$

where $v_f > 0$, $v_s > 0$, and

$$\begin{pmatrix} v_f^2 \\ v_s^2 \end{pmatrix} = \frac{V_A^2 + c_s^2}{2}\left[1 \pm \left(1 - \frac{4V_A^2 c_s^2\cos^2\theta}{\left(V_A^2 + c_s^2\right)^2}\right)^{1/2}\right]. \tag{61}$$

Here the plus sign corresponds to v_f^2 and the minus sign corresponds to v_s^2 and the subscripts f and s denote fast and slow, respectively. Because $|\theta| \neq 0, \pi, \pi/2$ the eigenvalues are all real, distinct, and nonzero so that the corresponding eigenvectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, and $\boldsymbol{e}_4$ are linearly independent. If these four column vectors are arranged into a matrix $P = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)$, then

$$AP = PD \tag{62}$$

where D is a diagonal matrix with the eigenvalues (λ_1, λ_2, λ_3, λ_4) along the main diagonal. The matrix P obtained using the nonnormalized eigenvectors can be written

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\dfrac{v_f^2 - c_s^2}{v_f^2\tan\theta} & -\dfrac{v_f^2 - c_s^2}{v_f^2\tan\theta} & -\dfrac{v_s^2 - c_s^2}{v_s^2\tan\theta} & -\dfrac{v_s^2 - c_s^2}{v_s^2\tan\theta} \\ \dfrac{v_f^2 - c_s^2}{v_f V_A\sin\theta} & -\dfrac{v_f^2 - c_s^2}{v_f V_A\sin\theta} & \dfrac{v_s^2 - c_s^2}{v_s V_A\sin\theta} & -\dfrac{v_s^2 - c_s^2}{v_s V_A\sin\theta} \\ c_s/v_f & -c_s/v_f & c_s/v_s & -c_s/v_s \end{bmatrix}. \tag{63}$$

[23] Introducing the new variable $\boldsymbol{u} = P^{-1}\boldsymbol{v}$, the linear system (58) takes the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + D\frac{\partial \boldsymbol{u}}{\partial x} = 0. \tag{64}$$

This system of four decoupled equations has the general solution

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{bmatrix} f_1\left(t - \dfrac{x}{v_f}\right) \\ g_1\left(t + \dfrac{x}{v_f}\right) \\ f_2\left(t - \dfrac{x}{v_s}\right) \\ g_2\left(t + \dfrac{x}{v_s}\right) \end{bmatrix}, \tag{65}$$

where f_1, f_2, g_1, and g_2 are arbitrary functions. Thus the solution of the original system (58) is given by $\boldsymbol{v} = P\boldsymbol{u}$ or

$$v_x(x, t) = f_1 + g_1 + f_2 + g_2, \tag{66}$$

$$v_z(x, t) = -\frac{v_f^2 - c_s^2}{v_f^2\tan\theta}(f_1 + g_1) - \frac{v_s^2 - c_s^2}{v_s^2\tan\theta}(f_2 + g_2), \tag{67}$$

$$b_z(x, t) = \frac{v_f^2 - c_s^2}{v_f V_A\sin\theta}(f_1 - g_1) + \frac{v_s^2 - c_s^2}{v_s V_A\sin\theta}(f_2 - g_2), \tag{68}$$

$$\frac{c_s\rho(x, t)}{\rho_0} = \frac{c_s}{v_f}(f_1 - g_1) + \frac{c_s}{v_s}(f_2 - g_2), \tag{69}$$

where the functions f_1, f_2, g_1, and g_2 have the arguments shown in equation (65). By inspection, one can immediately see that the forward propagating fast wave components are given by the terms proportional to f_1, the backward propagating fast wave components are given by the terms proportional to g_1, etc.

[24] If the components v_x, v_z, b_z, and ρ are all measured at $\boldsymbol{x} = 0$, say, then one may determine the functions f_j and g_j by the relation

$$\begin{bmatrix} f_1(t) \\ g_1(t) \\ f_2(t) \\ g_2(t) \end{bmatrix} = P^{-1}\begin{bmatrix} v_x(0, t) \\ v_z(0, t) \\ b_z(0, t) \\ \dfrac{c_s\rho(0, t)}{\rho_0} \end{bmatrix} = P^{-1}\boldsymbol{v}(0, t). \tag{70}$$

The complete wavefield is then determined by simply evaluating the functions f_1, f_2, g_1, and g_2 at the space-time arguments shown in equation (65).

[25] In terms of the transformed variable $\boldsymbol{u}$, the forward propagating component of the fast wave is given by

$$\boldsymbol{u}_+^f(x, t) = M_+^f\boldsymbol{u}(x, t) = \begin{bmatrix} f_1\left(t - \dfrac{x}{v_f}\right) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{71}$$

where

$$M_+^f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{72}$$

In terms of the original variables this becomes

$$\boldsymbol{v}_+^f(x,t) = PM_+^f P^{-1}\boldsymbol{v}(x,t) = P_+^f \boldsymbol{v}(x,t), \tag{73}$$

where $P_+^f = PM_+^f P^{-1}$ is the projection operator. The backward propagating component of the fast wave is given by

$$\boldsymbol{u}_-^f(x,t) = M_-^f \boldsymbol{u}(x,t) = \begin{bmatrix} 0 \\ g_1\left(t + \dfrac{x}{v_f}\right) \\ 0 \\ 0 \end{bmatrix}, \tag{74}$$

where

$$M_-^f = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{75}$$

or, in terms of the original variables,

$$\boldsymbol{v}_-^f(x,t) = PM_-^f P^{-1}\boldsymbol{v}(x,t) = P_-^f \boldsymbol{v}(x,t), \tag{76}$$

where $P_-^f = PM_-^f P^{-1}$ is the projection operator.

[26] The forward propagating component of the slow wave is given by

$$\boldsymbol{u}_+^s(x,t) = M_+^s \boldsymbol{u}(x,t) = \begin{bmatrix} 0 \\ 0 \\ f_2\left(t - \dfrac{x}{v_s}\right) \\ 0 \end{bmatrix}, \tag{77}$$

where

$$M_+^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{78}$$

or, in terms of the original variables,

$$\boldsymbol{v}_+^s(x,t) = PM_+^s P^{-1}\boldsymbol{v}(x,t) = P_+^s \boldsymbol{v}(x,t), \tag{79}$$

where $P_+^s = PM_+^s P^{-1}$ is the projection operator. The backward propagating component of the slow wave is given by

$$\boldsymbol{u}_-^s(x,t) = M_-^s \boldsymbol{u}(x,t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ g_2\left(t + \dfrac{x}{v_s}\right) \end{bmatrix}, \tag{80}$$

where

$$M_-^s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{81}$$

or, in terms of the original variables,

$$\boldsymbol{v}_-^s(x,t) = PM_-^s P^{-1}\boldsymbol{v}(x,t) = P_-^s \boldsymbol{v}(x,t), \tag{82}$$

where $P_-^s = PM_-^s P^{-1}$ is the projection operator.

[27] It only remains to derive explicit forms for the projection operators. Assume now that the eigenvectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{e}_4$ are normalized. Then $P^{-1} = P^T$ and one can show by straightforward computation that

$$P_+^f = PM_+^f P^T = \boldsymbol{e}_1\boldsymbol{e}_1^T, \tag{83}$$

$$P_-^f = PM_-^f P^T = \boldsymbol{e}_2\boldsymbol{e}_2^T, \tag{84}$$

$$P_+^s = PM_+^s P^T = \boldsymbol{e}_3\boldsymbol{e}_3^T, \tag{85}$$

$$P_-^s = PM_-^s P^T = \boldsymbol{e}_4\boldsymbol{e}_4^T. \tag{86}$$

These are the same projection operators obtained by *Glassmeier et al.* [1995]. However, the interpretation here is different. The wave decomposition derived here holds entirely in the space-time domain. Using this technique the data may be processed without Fourier transforms. However, the method is restricted to plane wave propagation in one direction only.

[28] The mode decomposition in the frequency domain is easily obtained from the mode decomposition in the space-time domain by Fourier transforming the various decomposition formulas with respect to time t and setting $x = 0$. The results can also be generalized to handle stochastic signals by assuming that the four different forward and backward propagating signal components are independent stochastic processes (or at least uncorrelated). Thus the power spectrum of $v_x(x, t)$ is the sum of the power spectra of $f_1(t)$, $g_1(t)$, $f_2(t)$, and $g_2(t)$; that is, the sum of the power spectrum of the forward propagating fast wave component plus the power spectrum of the backward propagating fast wave component plus the power spectrum of the forward propagating slow wave component plus the power spectrum of the backward propagating slow wave component.

4. Waves Propagating Perpendicular to $\boldsymbol{B}_0$

[29] For wave propagation perpendicular to $\boldsymbol{B}_0$ ($\theta = \pi/2$), the wave equations take the form

$$\frac{\partial v_x}{\partial t} = -\frac{c_s^2}{\rho_0}\frac{\partial \rho}{\partial x} - V_A\frac{\partial b_z}{\partial x}, \tag{87}$$

$$\frac{\partial b_z}{\partial t} = -V_A\frac{\partial v_x}{\partial x}, \tag{88}$$

$$\frac{\partial \rho}{\partial t} = -\rho_0\frac{\partial v_x}{\partial x}, \tag{89}$$

where $\boldsymbol{b}$ is expressed in Alfvén units and $b_x = 0$. The case $\theta = -\pi/2$ is obtained by making the replacement $V_A \to -V_A$. The general solution is given by

$$v_x(x,t) = f\left(t - \frac{x}{\lambda}\right) + g\left(t + \frac{x}{\lambda}\right), \tag{90}$$

$$b_z(x,t) = \frac{V_A}{\lambda}\left[f\left(t - \frac{x}{\lambda}\right) - g\left(t + \frac{x}{\lambda}\right)\right], \tag{91}$$

$$\frac{c_s\rho(x,t)}{\rho_0} = \frac{c_s}{\lambda}\left[f\left(t - \frac{x}{\lambda}\right) - g\left(t + \frac{x}{\lambda}\right)\right], \tag{92}$$

where f and g are arbitrary functions and $\lambda = (V_A^2 + c_s^2)^{1/2}$. By inspection, the forward propagating wave components are obtained by setting $g = 0$. The backward propagating wave components are obtained by setting $f = 0$. The solution is a fast wave because the slow wave and Alfvén wave are nonpropagating in the case $|\theta| = \pi/2$.

[30] The forward propagating wave components are given by

$$\boldsymbol{v}^+(\boldsymbol{x},t) = P^+\boldsymbol{v}(\boldsymbol{x},t), \tag{93}$$

where $\boldsymbol{v} = (v_x, b_z, c_s\rho/\rho_0)^T$ and

$$P^+ = \frac{1}{3}\begin{bmatrix} 1 & \lambda/V_A & \lambda/c_s \\ V_A/\lambda & 1 & V_A/c_s \\ c_s/\lambda & c_s/V_A & 1 \end{bmatrix}. \tag{94}$$

The backward propagating wave components are given by

$$\boldsymbol{v}^-(\boldsymbol{x},t) = P^-\boldsymbol{v}(\boldsymbol{x},t), \tag{95}$$

where

$$P^- = \frac{1}{3}\begin{bmatrix} 1 & -\lambda/V_A & -\lambda/c_s \\ -V_A/\lambda & 1 & V_A/c_s \\ -c_s/\lambda & c_s/V_A & 1 \end{bmatrix}. \tag{96}$$

These projection operators provide the desired decomposition. It should be mentioned that the projection operators are not unique in this case as can be seen from the fact that $\det(P^{\pm}) = 0$.

[31] As shown for the Alfvén wave modes in subsection 2.1, the mode decomposition procedure takes the same form in the Fourier domain as in the space-time domain. The projection relations are obtained by setting $x = 0$ in equations (93) and (95) and then taking the Fourier transform with respect to time. The generalization to random fields is accomplished as described in subsection 2.1.

5. Conclusions

[32] For MHD wavefields propagating in one direction only, it has been shown how the forward and backward propagating components of the fast, slow, and Alfvén modes can be extracted from measured data. The decomposition technique is formulated in the space-time domain and allows for real-time processing of the data without the use of Fourier transforms. Because the projection operators are matrices with elements that are all constants, the same projection operators can also be used to obtain the mode decomposition in the frequency domain. Therefore the decomposition procedure is the same in both the space-time domain and the frequency domain. The major limitation of the technique is that it only applies to wavefields with a unique propagation direction and, moreover, this direction must be known to perform the analysis.

[33] Thus far there has been no discussion of the important implementation issues that must be addressed before the space-time decomposition technique can be successfully implemented in practice. For example, it is necessary to determine the sensitivity of the method to the level of noise in the data and to quantify the error due to uncertainty in the wave propagation direction. Such issues lie outside the scope of this paper but should not present a significant impediment to the practical realization of the technique.

[34] The implementation of the space-time decomposition technique can in principle be applied to data from a single spacecraft if the propagation direction is known, either from a minimum variance analysis or by other means. However, the determination of the propagation direction or $\boldsymbol{k}$-vector is more reliably obtained using data from multiple spacecraft. The multiple spacecraft approach to finding the propagation direction is called "k-filtering" or the "generalized minimum variance analysis" and is based on the technique developed by *Pinçon and Lefeuvre* [1991] and *Motschmann et al.* [1995, 1996].

[35] The technique of k-filtering has been combined with the mode decomposition technique developed by *Glassmeier et al.* [1995] to implement the "wave telescope," a novel invention for identifying wave modes and propagation directions using data from fleets of multiple spacecraft such as ESA's Cluster mission [*Escoubet et al.*, 2001]. The basic idea of the wave telescope was developed by *Dunlop et al.* [1988], *Neubauer and Glassmeier* [1990], and *Neubauer et al.* [1990]. The theoretical framework for the analysis and interpretation of measurements from multiple spacecraft was worked out by *Pinçon and Lefeuvre* [1991], *Glassmeier et al.* [1995], and *Motschmann et al.* [1996]. Recent implementation of the k-filtering technique using multipoint spacecraft data has been reported by *Glassmeier et al.* [2001], *Narita et al.* [2003, 2004, 2006], and *Narita and Glassmeier* [2005]. In the special cases of wave propagation parallel and perpendicular to the background magnetic field the mode decompositions developed in this paper can usefully be applied in the wave telescope.

[36] It is of interest to briefly compare the practical utility of the space-time decomposition scheme presented here to the frequency/wave vector decomposition scheme developed by Glassmeier and others. Both methods are linear and apply only to linear (small-amplitude) wavefields. While the frequency/wave vector decomposition scheme can be applied to wavefields possessing multiple wave vectors, the space-time decomposition scheme cannot. This is the major drawback of the space-time decomposition scheme in its current form. The obvious advantage of the space-time decomposition scheme is that it can be performed in real time with a minimum of computational work, but for the ULF-VLF range of frequencies encountered in space physics applications the computational work required to perform

the Fourier transforms needed for the frequency/wave vector decomposition scheme does not present a significant drawback. Thus for space physics applications, the space-time decomposition scheme is not as versatile as the frequency/wave vector decomposition scheme. Nevertheless, it may still be useful in some niche applications.

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